Base Case (All Loads, Banking)

Introduction

Thus far, we have considered separately the effects of the vertical force W_T, turning forces T_L and T_R, road banking through an angle α, acceleration force A, braking force B and other loads such as Ballast, Driver, Fuel and Aerodynamic. We now will bring all these loads together and write the equations describing the system. At this stage all loads are present, the magnitude and location of the CGs for all loads are known, the geometry (track and wheel base) is set and all types of suspension components are included in the model.

Model Equations

[1] $P_1 + P_2 + P_3 + P_4 = W_T \cos\alpha + (T_L + T_R)\sin\alpha + (Ba + Dr + Fu)\cos\alpha + Ae$

[2] $P_1t_1 - P_2t_2 + P_3t_3 - P_4t_4 = [-R]\{(W_T \sin\alpha - (T_L + T_R)\cos\alpha)\, z_T - (W_T \cos\alpha + (T_L + T_R)\sin\alpha) y_T$

$(Ba\, z_B + Dr\, z_D + Fu\, z_F + Ae\, z_A)\sin\alpha - (Ba\, y_B + Dr\, y_D + Fu\, y_F)\cos\alpha +$

$Ae\, y_A\}$

[3] $P_1l_1 + P_2l_2 - P_3l_3 - P_4l_4 = (W_T \cos\alpha + (T_L + T_R)\sin\alpha)\, x_T$

$+ (B-A)\, z_T + (Ba\, x_B + Dr\, x_D + Fu\, x_F)\cos\alpha + Ae\, x_A$

[4] $P_1 - K_1V_1 - m_{R1}k_{RF}d_R = m_1k_1 (D_1 + m_1R_{C1}) + m_{R1}k_{RF} (D_{R1} + m_{R1}R_{C1}) + m_{S1}(f_{S1} + f_{SP1}) + w_{u1}$

[5] $P_2 - K_2V_2 + m_{R2}k_{RF}d_R = m_2k_2 (D_2 + m_2R_{C2}) + m_{R2}k_{RF} (D_{R2} + m_{R2}R_{C2}) + m_{S2}(f_{S2} + f_{SP2}) + w_{u2}$

[6] $P_3 - K_3V_3 - m_{R3}k_{RR}d_R = m_3k_3 (D_3 + m_3R_{C3}) + m_{R3}k_{RR} (D_{R3} + m_{R3}R_{C3}) + m_{S3}(f_{S3} + f_{SP3}) + w_{u3}$

[7] $P_4 - K_4V_4 + m_{R4}k_{RR}d_R = m_4k_4 (D_4 + m_4R_{C4}) + m_{R4}k_{RR} (D_{R4} + m_{R4}R_{C4}) + m_{S4}(f_{S4} + f_{SP4}) + w_{u4}$

[8] $C_1 V_1 + C_2 V_2 + C_3 V_3 + C_4 V_4 = 0$

[9] $V_1 - V_2 - G\, d_R = 0$

Matrix Formulation

As shown previously, this system of equations can be written in matrix form, **A** X = **B**, with

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ t_1 & -t_2 & t_3 & -t_4 & 0 & 0 & 0 & 0 & 0 \\ l_1 & l_2 & -l_3 & -l_4 & 0 & 0 & 0 & 0 & 0 \\ E_1 & 0 & 0 & 0 & -K_1 & 0 & 0 & 0 & K_{R1} \\ 0 & E_2 & 0 & 0 & 0 & -K_2 & 0 & 0 & K_{R2} \\ 0 & 0 & E_3 & 0 & 0 & 0 & -K_3 & 0 & K_{R3} \\ 0 & 0 & 0 & E_4 & 0 & 0 & 0 & -K_4 & K_{R4} \\ 0 & 0 & 0 & 0 & C_1 & C_2 & C_3 & C_4 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -G \end{bmatrix}$$

where

$E_i = 1$ for i = 1 … 4

$K_i = m_i^2\, k_i$ for i = 1 … 4

$K_{R1} = -m_{R1}k_{RF}$ $K_{R2} = m_{R2}k_{RF}$ $K_{R3} = -m_{R3}k_{RR}$ $K_{R4} = m_{R4}k_{RR}$

C_i = Compatibility coefficients

$$G = \frac{y_1 + y_2}{y_{R1} + y_{R2}}$$

$$X = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ d_R \end{Bmatrix} \quad \text{and} \quad B = \begin{Bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \\ B_7 \\ B_8 \\ B_9 \end{Bmatrix}$$

where

$B_1 = W_T \text{ Cos } \alpha + (T_L + T_R) \text{ Sin } \alpha + (Ba + Dr + Fu) \text{ Cos } \alpha + Ae$

$B_2 = [-R]\{(W_T \text{ Sin } \alpha - (T_L + T_R) \text{ Cos } \alpha)\, z_T - (W_T \text{ Cos } \alpha + (T_L + T_R) \text{ Sin } \alpha)\, y_T$
$(Ba\, z_B + Dr\, z_D + Fu\, z_F + Ae\, z_A) \text{ Sin } \alpha - (Ba\, y_B + Dr\, y_D + Fu\, y_F) \text{ Cos } \alpha + Ae\, y_A\}$

$B_3 = (W_T \text{ Cos } \alpha + (T_L + T_R) \text{ Sin } \alpha)\, x_T + (B-A)\, z_T + (Ba\, x_B + Dr\, x_D + Fu\, x_F) \text{ Cos } \alpha + Ae\, y_A$

$B_4 = m_1k_1 (D_1 + m_1R_{C1}) + m_{R1}k_{RF} (D_{R1} + m_{R1}R_{C1}) + m_{S1}(f_{S1} + f_{SP1}) + w_{u1}$

$B_5 = m_2k_2 (D_2 + m_2R_{C2}) + m_{R2}k_{RF} (D_{R2} + m_{R2}R_{C2}) + m_{S2}(f_{S2} + f_{SP2}) + w_{u2}$

$B_6 = m_3k_3 (D_3 + m_3R_{C3}) + m_{R3}k_{RR} (D_{R3} + m_{R3}R_{C3}) + m_{S3}(f_{S3} + f_{SP3}) + w_{u3}$

$B_7 = m_4k_4 (D_4 + m_4R_{C4}) + m_{R4}k_{RR} (D_{R4} + m_{R4}R_{C4}) + m_{S4}(f_{S4} + f_{SP4}) + w_{u4}$

$B_8 = 0$

$B_9 = 0$

[–R] → Use negative sign for right turn

Solving this system of equations, $X = \mathbf{A}^{-1} \mathbf{B}$, results in 4 wheel loads, 4 spring deformations and one roll bar distortion.

$P_1, P_2, P_3, P_4, V_1, V_2, V_3, V_4, d_R$

The wheel loads are used to calculate left, rear and wedge bias (B_L, B_R, B_W).

$$B_L = \frac{P_1 + P_3}{W_T} \qquad B_R = \frac{P_3 + P_4}{W_T} \qquad B_w = \frac{P_2 + P_3}{W_T}$$

With known geometry (spring co-ordinates x and y), the wheel motions are used to determine the co-ordinates of midpoints A, B, C and D.

Mid-Point Co-ordinates

	A	**B**	**C**	**D**
x	$\frac{y_1x_2 + y_2x_1}{y_1 + y_2}$	0	$-\frac{y_3x_4 + y_4x_3}{y_3 + y_4}$	0
y	0	$-\frac{x_2y_4 + x_4y_2}{x_2 + x_4}$	0	$\frac{x_1y_3 + x_3y_1}{x_1 + x_3}$
z	$\frac{y_1V_2 + y_2V_1}{y_1 + y_2}$	$\frac{x_2V_4 + x_4V_2}{x_2 + x_4}$	$\frac{y_3V_4 + y_4V_3}{y_3 + y_4}$	$\frac{x_1V_3 + x_3V_1}{x_1 + x_3}$

Note: x_i and y_i are the co-ordinates of tire contact points and V_i are the vertical wheel movements

These midpoint co-ordinates will be used to find the equation for the plane of the wheel locations.

$$AAx + BBy + CCz + DD = 0$$

Where

$$AA = y_1 V_2 - y_2 V_1 + y_C (V_1 - V_2) - z_C(y_1 - y_2)$$

$$BB = - x_1V_2 + x_2V_1 - x_C (V_1 - V_2) + z_C(V_1 - V_2)$$

$$CC = x_1y_2 - x_2y_1 + x_C (y_1 - y_2) + y_C(x_2 - x_1)$$

$$DD = x_C (-y_1d_2 + y_2d_1) + y_C (x_1d_2 - x_2 d_1) + z_C (y_1 x_2 - y_2 x_1)$$

The equation of the plane is used to determine ride changes at the ride points. We substitute the co-ordinates x_r and y_r for each ride point into the equation of the plane to find its change in elevation, z_r.

$$z_r = \frac{-DD - AAx_r - BBy_r}{CC}$$

Wheel Motions are also used to find vertical displacements at the wheel midpoints, d_A, d_B, d_C, d_D, for a given geometry, and then to determine the chassis motion components: squat, pitch and roll. First, we find the change in ride for the origin (x = 0 and y = 0).

$$d_0 = z_0 = \frac{-DD - AA(0) - BB(0)}{CC} = -\frac{DD}{CC}$$

Next we use the change in elevation for midpoints A and B, d_A and d_B, to find the motion components

Squat → V_0 *Pitch* → $p = \frac{d_A - d_0}{x_A}$ *Roll* → $r = \frac{d_B - d_0}{y_B}$

Anti Forces

Review

Under the action of vertical (W_T, Ba, Dr, Fu and Ae), lateral (T_L or T_R) or longitudinal (A and B) forces, the chassis transmits loads to the contact patch under the wheels through the suspension components such as: springs, roll bars, dampers, track bars and other links.

Springs and roll bars are flexible and elastic components that must deform before they can transmit the loads from the chassis. The amount of deformation depends on their stiffness or ‘spring rate’. Dampers or shock absorbers are flexible, visco-elastic components, which means that their deformation is time-dependent. The force in a damper depends on the velocity of the applied deformation. Dampers also normally have some level of preload, which is a force and is not time dependent. Track bars and other links have very high stiffness and will be treated as rigid elements with very small, negligible, deformations under load.

The springs are always present in the suspension of a racing car. The other suspension components participate in transmitting loads, from the chassis to the wheels, at varying proportions depending on the material characteristics of the components (spring rate for the roll bar, shaft speed for the dampers) and the geometry of the suspension (location of Instant Centre, IC).

When two or more suspension components are present and active, *e.g.*

- spring + roll bar
- spring + damper
- spring + roll bar + damper

the total force, required for equilibrium, that is passed from the chassis to the wheels is split between the elements participating in the mechanism in proportion to the stiffness and geometry of the components.

If there are no other suspension components, the springs pass the total load from the chassis to the wheel. If there are active suspension components, they will take some of the total load and the spring will take the rest. This smaller spring load results in smaller spring deformations and lower values for chassis motion (squat, pitch and roll).

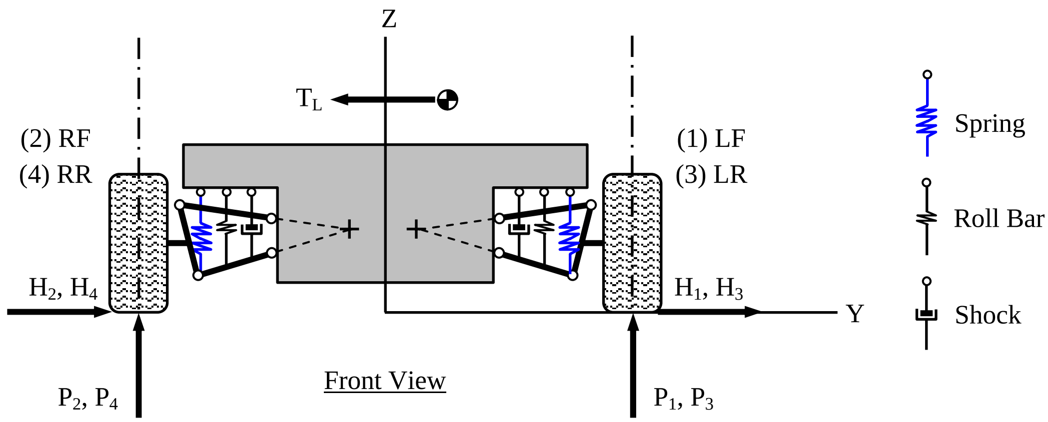
The amount of load the suspension elements are capable of developing depends on the geometry of the components, motion ratios, spring rates and location of the Instant Centre, IC.

Independent suspensions can be used at front or rear corners. For the purpose of this book, live axle track bar suspensions are used only at the rear. Roll bars are used at the front and/or rear. Dampers can also be used front and rear. Both independent and track bar suspensions are used in combination with roll bars and dampers. Springs are always present at all corners in all suspension mechanisms.

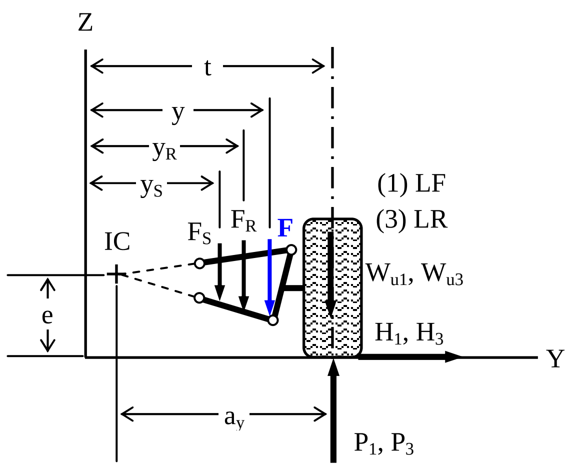
Independent Suspension

Under an independent suspension system, the horizontal lateral force induced by a turn produces a vertical reaction at the wheel contact patch, P, and a horizontal reaction H as shown. A portion of the total wheel load is passed to the chassis through the deformable components (spring, roll bar, shocks) and the rest is passed through the rigid links (upper and lower arms).

Left Turn



Corners (1) and (3). Left Turn



The horizontal reaction, H, to the turning force, T_L, depends on the portion contributed by each tire. The total $H_1 + H_2$ acting on the front tires is proportional to the amount of mass on the front tires. This portion can be divided between the front tires by assuming each contributes in proportion to the wheel load on the corner.

This can not be expressed in a linear equation so we assume the entire turning force is distributed in proportion to the total wheel load. This assumption is inaccurate when there is a longitudinal force or aerodynamic load.

$$H = \frac{P}{W_T} T_L \qquad \rightarrow \qquad H = \frac{T_L}{W_T} P$$

Equilibrium

$\boxed{\Sigma F_Z = 0}$ $\qquad P - mF - m_R F_R - m_S F_S - w_u = 0$

$\boxed{\Sigma M_{IC} = 0}$

$$P a_y + H\,e - mF a_y - m_R F_R a_y - m_S F_S a_y - w_u\,a_y = 0$$

dividing by ‘a_y’

$$P + H\,\frac{e}{a_y} - mF - m_R F_R - m_S F_S - w_u = 0$$

Spring, roll bar and damper behaviour

$$F = k\,(d + D + m_y R_C) \qquad F_R = k_R\,(d_R + D_R + m_R R_C) \qquad F_S = f_S + f_{SP}$$

Back substitution

$$P + \frac{T_L}{W_T} P \frac{e}{a_y} - m\,k\,(d + D + mR_C) - m_R\,k_R\left(\frac{m_R}{m} d + D_R\right) - m_S\,(f_S + f_{SP}) - w_u = 0$$

re-arranging and collecting terms

$$P\,[1 + \frac{T_L}{W_T}\,\frac{e}{a_y} - mkd - m_R k_R d_R = mk(D + mR_C) + m_R k_R\,(D_R + m_R\,R_C) + m_S F_S + w_u$$

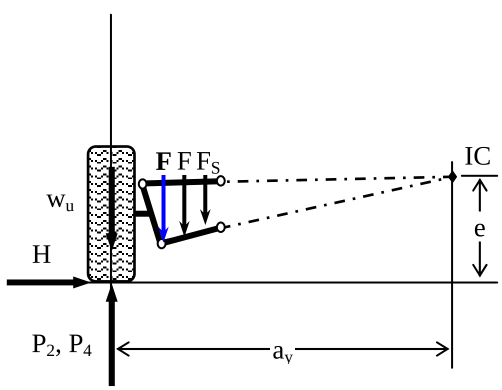
$$P\,[1 + \frac{T_L}{W_T}\,\frac{e}{a_y} - KV - m_R k_R d_R = mk(D + mR_C) + m_R k_R\,(D_R + m_R\,R_C) + m_S F_S + w_u$$

Individual corners

$$P_1\,[1 + \frac{T_L}{W_T}\left(\frac{e}{a_Y}\right)_1] - K_1 V_1 - m_{R1} k_{RF} d_R = m_1 k_1\,(D_1 + m_1 R_{C1}) + m_{R1} k_{RF}\,(D_{R1} + m_{R1}\,R_{C1}) + m_{S1} F_{S1} + w_u$$

$$P_3\,[1 + \frac{T_L}{W_T}\left(\frac{e}{a_Y}\right)_3] - K_3 V_3 - m_{R3} k_{RR} d_R = m_3 k_3\,(D_3 + m_3 R_{C3}) + m_{R2} k_{RR}\,(D_{R3} + m_{R3}\,R_{C3}) + m_{S3} F_{S3} + w_u$$

Corners (2) and (4). Left Turn



Equilibrium

$\boxed{\Sigma M_{IC} = 0}$

$$-Pa_y + H\,e + mFa_y + m_R F_R a_y + m_S F_S a_y + w_u\,a_y = 0$$

dividing by 'a_y'

$$-P + H\,\frac{e}{a_y} + mF + m_R F_R + m_S F_S + w_u = 0$$

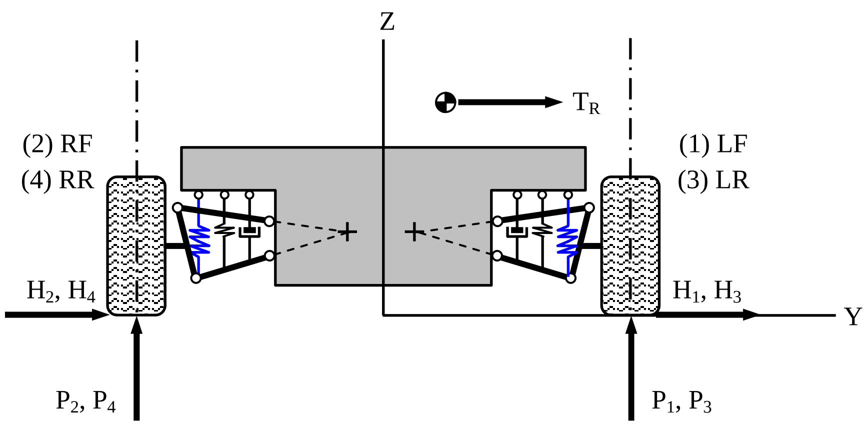
Using a similar algebraic procedure as for the left corner

Individual corners

$$P_2\left[1 - \frac{T_L}{W_T}\left(\frac{e}{a_y}\right)_2\right] - K_2V_2 - m_{R2}k_{RF}d_R = m_2k_2\,(D_2 + m_2R_{C2}) + m_{R2}k_{RF}\,(D_{R2} + m_{R2}\,R_{C2}) + m_{S2}F_{S2}$$

$$P_4\left[1 - \frac{T_L}{W_T}\left(\frac{e}{a_y}\right)_4\right] - K_4V_4 - m_{R4}k_{RR}d_R = m_4k_4\,(D_4 + m_4R_{C4}) + m_{R4}k_{RR}\,(D_{R4} + m_{R4}\,R_{C4}) + m_{S4}F_{S4}$$

Right Turn



Following the same steps as for the left turn case, with due to consideration to the change in sign of the horizontal forces, we obtain the following:

$$P_1\left[1 + \frac{T_R}{W_T}\left(\frac{e}{a_Y}\right)_1\right] - K_1V_1 - m_{R1}k_{RF}d_R = m_1k_1\,(D_1 + m_1R_{C1}) + m_{R1}k_{RF}\,(D_{R1} + m_{R1}\,R_{C1}) + m_{S1}F_{S1} + w_{u1}$$

$$P_3\left[1 + \frac{T_R}{W_T}\left(\frac{e}{a_y}\right)_3\right] - K_3V_3 - m_{R3}k_{RR}d_R = m_3k_3\,(D_3 + m_3R_{C3}) + m_{R2}k_{RR}\,(D_{R3} + m_{R3}\,R_{C3}) + m_{S3}F_{S3} + w_{u3}$$

$$P_2\left[1 - \frac{T_R}{W_T}\left(\frac{e}{a_y}\right)_2\right] - K_2V_2 - m_{R2}k_{RF}d_R = m_2k_2\,(D_2 + m_2R_{C2}) + m_{R2}k_{RF}\,(D_{R2} + m_{R2}\,R_{C2}) + m_{S2}F_{S2} + w_{u2}$$

$$P_4\left[1 - \frac{T_R}{W_T}\left(\frac{e}{a_y}\right)_4\right] - K_4V_4 - m_{R4}k_{RR}d_R = m_4k_4\,(D_4 + m_4R_{C4}) + m_{R4}k_{RR}\,(D_{R4} + m_{R4}\,R_{C4}) + m_{S4}F_{S4} + w_{u4}$$

Simplified Equations

These equations correspond to Equations [4] through [7] for the Base Case discussed before.

$$P_1 \left[1 + \frac{T_L + T_R}{W_T} \left(\frac{e}{a_Y}\right)_1 \right] - K_1V_1 - m_{R1}k_{RF}d_R = m_1k_1 (D_1 + m_1R_{C1}) + m_{R1}k_{RF} (D_{R1} + m_{R1} R_{C1}) + m_{S1}F_{S1} + w_{u1}$$

$$P_2 \left[1 - \frac{T_L + T_R}{W_T} \left(\frac{e}{a_y}\right)_2 \right] - K_2V_2 - m_{R2}k_{RF}d_R = m_2k_2 (D_2 + m_2R_{C2}) + m_{R2}k_{RF} (D_{R2} + m_{R2} R_{C2}) + m_{S2}F_{S2} + w_{u2}$$

$$P_3 \left[1 + \frac{T_L + T_R}{W_T} \left(\frac{e}{a_y}\right)_3 \right] - K_3V_3 - m_{R3}k_{RR}d_R = m_3k_3 (D_3 + m_3R_{C3}) + m_{R2}k_{RR} (D_{R3} + m_{R3} R_{C3}) + m_{S3}F_{S3} + w_{u3}$$

$$P_4 \left[1 - \frac{T_L + T_R}{W_T} \left(\frac{e}{a_y}\right)_4 \right] - K_4V_4 - m_{R4}k_{RR}d_R = m_4k_4 (D_4 + m_4R_{C4}) + m_{R4}k_{RR} (D_{R4} + m_{R4} R_{C4}) + m_{S4}F_{S4} + w_{u4}$$